

Eccentric Motion of Spinning Compact Binaries

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The equations of motion for spinning compact binaries on eccentric orbits are treated perturbatively in powers of a fractional mass-difference ordering parameter. The solution is valid through first order in the mass-difference parameter. A canonical point transformation removes the leading order terms of the spin-orbit Hamiltonian which induce a wiggling precession of the orbital angular momentum around the conserved total angular momentum, a precession which disappears in the case of equal masses or one single spin. Action-angle variables are applied which make a canonical perturbation theory easily treatable.

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I. INTRODUCTION

Compact binary star systems are often investigated in general relativity when moving on orbits with zero eccentricity. This is usually justified by the circularisation effect due to radiation reaction for *isolated* binaries [1–6] which becomes rather strong in the late stages of the binary’s life. For this reason, numerical relativity simulations of compact binaries

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(which typically model the late inspiral phase) often start from quasi-circular orbits [7]. Nonetheless, binaries can retain finite eccentricity through various mechanisms involving either additional bodies or gaseous environments [8]. Eccentric binaries typically lead to enhanced and more complex gravitational wave (GW) emission compared to the quasi-circular case [9, 10] which leads to two consequences: (i) Eccentric binaries can be detected out to larger distances (up to two orders of magnitude in detection volume for adLIGO [9]) than quasi-circular binaries (everything else being equal), which affects their (poorly known) event rates [8]. (ii) Parameter estimation for GW detectors typically adopt quasi-circular templates which can severely limit the ability to detect GWs and to recover source parameters [9, 11].

In order to also model the GW signals from *eccentric* binaries reliably, it is essential to include higher order general relativistic effects that are very well described within the post-Newtonian framework, also see the discussion in Ref. [12]. The reason for it is that spin precession and periastron advance will generate modifications (e.g. side-bands) in the GW Fourier domain. If they are not included, the correlation of real detector data with incomplete and therefore non-optimal GW templates leads to computed system parameters that are displaced with respect to "real" ones, although the signal may be covered more or less effectively for special configurations.

The point-mass contributions to the post-Newtonian Hamiltonians in the Arnowitt-Deser-Misner (ADM) gauge have been computed through the fourth post-Newtonian (4PN) order ($=\mathcal{O}(v/c)^8$, where v is a typical internal velocity and c the speed of light) in Ref. [13] (the 3PN calculation can be found in an earlier publication [14]). The spin-orbit contributions are derived in Refs. [15] and [16] through next-to-leading order ($=\text{NLO}$) and in Ref. [17] through next-to-next-to-leading order ($=\text{NNLO}$) for compact binaries. NNLO effects in the spin-orbit coupling for an arbitrary number n of compact spinning objects have been derived in [18].

The Hamiltonian prescription leads to a number of evolution equations for the radial part of the binary motion and for quantities being related to the orientations of the spins and the orbital plane. The solution to the linear-in-spin problem for compact binary systems has been discussed frequently in the recent years, see below. For example, in [19], the circular-orbit motion has been solved with the help of a sequence of Lie transformations. In other publications, see [20], a Kepler equation for compact binaries with spin has been given; in [21] the GW forms of eccentric binaries with spin were worked out – the equations of motion of the entering spin orientation angles are given, but not solved there. In [22], tail-induced spin-orbit effects in the energy flux and the GW forms have been derived for circular orbits with arbitrary masses.

Summarising, in the current article we generalise the recent analytic results that are known (we omit the included spin and PN orders here), for

- circular orbits, arbitrary spins and masses [23–28],
- eccentric orbits, arbitrary spins, but equal masses [29],
- eccentric orbits, one single spin, and arbitrary masses [29],
- eccentric orbits, aligned spins and arbitrary masses [30, 31],
- circular orbits, arbitrary spin orientations, allowing slightly unequal masses [19, 32],

and references therein, to **eccentric orbits, arbitrary spins and allowing slightly unequal masses**. For a first insight, we will include the gravitational leading-order spin-orbit coupling and the second post-Newtonian (2PN) point-mass (PM) interaction Hamiltonian for compact binaries [33, 34].

Spin(a)-spin(b) and spin-squared couplings also have to be included at some instant of time in their orbital evolution, especially at the late stage of the inspiral, but we disregard them in this article for two reasons. The first one is that they turn out to be small compared to the other terms considered here, at least at large binary separations where the orbital angular momentum is much larger than the spin. The second reason is that they cannot be regarded as a contribution that is growing with the mass difference and, for equal masses, a closed-form solution for precession including those terms is not known until now.

Our tool will be the application of action and angle (AA) variables (see our subsection IV A and Refs. [35, 36] for their definitions and applications) and a subsequent generalisation of the Delaunay variables (see e.g. [37]) which are derived from the AA variables.¹ The generalisation will be performed in three steps: (i) taking the expression of the interaction Hamiltonian in terms of the variables provided in [19], (ii) computing the action-angle variables from those expressions, and (iii) eliminating the degeneracy at Newtonian level, leading to the Delaunay-type variables with spin,

¹ Those are related to the recently introduced “Hill - inspired” variables (see [38] for reasons of this terminology) for compact binaries with spin.

using the definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.² In this context we like to mention a perturbative treatment of star resonances in Newtonian orbits by [36], where also action-angle variables came to application to characterise the zero-order problem where no oscillations take place.

At this point, we like to state why we prefer to work with canonical variables. Their advantage is to make canonical perturbation theory feasible. Although it is not a necessity to tackle perturbation problems in this way [42], it makes the calculation more practical because standard Poisson brackets remain valid to obtain the EOM (equations of motion) after any transformation.

The article is organised as follows. In Section II we provide the Hamiltonian interaction terms. In Section III we discuss the known solution to the problem of unperturbed Hamiltonian equations of motion for eccentric binaries with spin-orbit interaction. In Section IV we summarise the main aspects of the Hamilton-Jacobi (HJ) theory which is used to solve the perturbed equations of motion in a specific manner. We summarise the definition of action-angle variables for librational motion and we present some techniques for the radial part of the generating function and also those for the elimination of degeneracy conditions in the resulting equations of motion. In a subsequent Section V we present our main result: the application of the HJ perturbation theory to the case of eccentric binaries with spin, where we expand the solution to the first order of the mass difference function ϵ .

II. INCLUDED INTERACTION TERMS

The point-mass Hamiltonian (subscript “PM”) to second post-Newtonian accuracy will be given below. The symbols are explained in Table I. We work in dimension-less units as given in Eqs. (6)-(9) of [30] to obtain Eqs. (2.1-2.4) below, with the only exception of additionally imposing fast-spinning components, for convenience of the reader³. We set $c=1$, but retain c as a power-counting parameter in order to easily plug in numbers for explicit examples.

$$H_{\text{PM}}^{\text{N}} = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - \frac{1}{r}, \quad (2.1)$$

$$H_{\text{PM}}^{1\text{PN}} = c^{-2} \left\{ \frac{L^4(3\eta - 1)}{8r^4} - \frac{L^2(\eta + 3)}{2r^3} + \frac{1}{2r^2} + p_r^2 \left(\frac{(3\eta - 1)L^2}{4r^2} - \frac{2\eta + 3}{2r} \right) + \frac{1}{8}(3\eta - 1)p_r^4 \right\}, \quad (2.2)$$

$$\begin{aligned} H_{\text{PM}}^{2\text{PN}} = c^{-4} & \left\{ \frac{L^6(5(\eta - 1)\eta + 1)}{16r^6} - \frac{L^4(3\eta^2 + 20\eta - 5)}{8r^5} + \frac{L^2(8\eta + 5)}{2r^4} + \frac{-3\eta - 1}{4r^3} \right. \\ & + p_r^6 \frac{1}{16}(5(\eta - 1)\eta + 1) + p_r^4 \left(\frac{-8\eta^2 - 20\eta + 5}{8r} + \frac{3(5(\eta - 1)\eta + 1)L^2}{16r^2} \right) \\ & \left. + p_r^2 \left(\frac{(-4\eta^2 - 20\eta + 5)L^2}{4r^3} + \frac{3(5(\eta - 1)\eta + 1)L^4}{16r^4} + \frac{11\eta + 5}{2r^2} \right) \right\}. \end{aligned} \quad (2.3)$$

The terms linear in spin through leading order [15] read, “SO” denoting spin-orbit coupling and “LO” leading order,

$$H_{\text{LO}}^{\text{SO}} = \frac{c^{-3}}{4r^3} \left\{ (2\eta + 3\sqrt{1 - 4\eta} + 3) (\mathbf{L} \cdot \mathbf{S}_1) + (2\eta - 3\sqrt{1 - 4\eta} + 3) (\mathbf{L} \cdot \mathbf{S}_2) \right\}. \quad (2.4)$$

Those Hamiltonians generate equations of motion that, currently, can be solved only in a perturbative manner. One can construct a more practical set of spin variables that distinguish “constant” from “oscillatory” (the term “constant” is equal to “integral of motion” and “oscillatory” is equal to “give zero time average”; both are meant in a context that we will explain later on in Subsec. IV B) contributions. We will give the Hamiltonian in these new coordinates in Section IV. Let us first turn to the known solution for binaries of equal masses (also including the single-spin case) – which will serve as a basis for our calculation.

² This canonical definition was not used within the Hill variables, see [39]. For reasons of current research in a slightly different context, also concerning the discussion in a current article [40], we like to give reference to the publication of Gurfil et al. [41], dealing with a distinction of the usage of $\mathbf{L}_N := \mathbf{r} \times \dot{\mathbf{r}}$ which is not a canonical quantity. Note that the used variables do *not* diverge for small azimuthal angles Θ as stated by those authors – in fact, they degenerate in the exact $\Theta = 0$ case.

³ The LO SO interaction is formally of 1PN order. Imposing fast-spinning components, it is shifted to 1.5PN order, slow rotation shifts it further to 2PN order.

III. SOLUTION TO THE ECCENTRIC SPIN-ORBIT PROBLEM AT LEADING ORDER WITH EQUAL MASSES

We define the orbital plane to be that plane which is moving perpendicularly to the canonical orbital angular momentum \mathbf{L} . The motion of compact binaries in the orbital plane can be prescribed by the following system of equations, which uses definitions of several orbital elements to be found in Table I:

$$r = a_r (1 - e_r \cos \mathcal{E}), \quad (3.1a)$$

$$\phi = 2 \arctan \left\{ \sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan \frac{\mathcal{E}}{2} \right\} + \mathcal{O}(c^{-4}), \quad (3.1b)$$

$$\mathcal{M} = \mathcal{E} - e_t \sin \mathcal{E} + \mathcal{O}(c^{-4}). \quad (3.1c)$$

The geometrical meaning of the above relations (at Newtonian level) may be found, for example in Colwell's book [43]. Their derivation is given in, e. g., [44] and for the aligned-spin case in [31], including the energy and angular momentum decay due to radiation reaction. For the leading-order spin-orbit case with (i), single spin or (ii), equal masses, the above terms get spin-orientation corrections, see [29], and the following orientation equation

$$\Upsilon - \Upsilon_0 = \frac{\chi_{so} J}{c^3 L^3} (\phi + e \sin \phi), \quad (3.2)$$

has to be added to prescribe the full conservative motion of the system. In Eq. (3.2), χ_{so} is a term that is either equal to 7/8 for the equal-mass case or equal to a function of the masses in the single-spin case (see their Eqs. (2.5a, 2.5b) and (4.36)), the angle Υ is the canonical coordinate conjugate to the “momentum” J and ϕ_S the one associated to S_G . The solution to ϕ_S may also be given, but is irrelevant in that case because it enters nowhere explicitly.

Quantity	Description
c^{-1} Power counting for post-Newtonian orders
$n\text{PN}$ n^{th} post-Newtonian order, $\mathcal{O}(c^{-2n})$
η Symmetric mass ratio: $\eta := m_1 m_2 / (m_1 + m_2)^2$
$ E $ Absolute value of binding energy
L Angular momentum of orbit, $L := \mathbf{L} $
\mathcal{M}, ℓ_D Mean anomaly
\mathcal{N}	Mean motion or radial angular velocity, respectively
\mathcal{E} Eccentric anomaly
\mathbf{p} Linear momentum: $p := \mathbf{p} $, $p_r := (\mathbf{n}_{12} \cdot \mathbf{p})$
\mathbf{r} Radial separation: $r := \mathbf{r} $, $\mathbf{n}_{12} := \mathbf{r}/r$
S_G Total spin: $S_G := \mathbf{S}_1 + \mathbf{S}_2 $
ϕ Orbital phase, measured from the pericenter
ϕ_S Spin phase, see Fig. 1 of [32]
Φ Total orbital phase in one radial period
Υ	... Rotation angle for \mathbf{L} around fixed unit vector \mathbf{e}_Z
a_r Semimajor axis
e_r Radial eccentricity
e_ϕ Phase eccentricity
e_t Time eccentricity
ϵ Mass difference function: $\epsilon^2 := \frac{1}{4} - \eta$

TABLE I: Shorthands of quantities frequently used in this article. The Poisson brackets for the coordinates and momenta are to be taken from [19].

The quantities e_r, a_r and so on essentially depend on the included interaction terms. This parameterisation will enter, at its leading order, the solution to the perturbed motion as basis. It will be applied to the HJ theory (which is in fact standard), summarised in the subsequent section for convenience of the reader.

IV. HAMILTON-JACOBI THEORY

The Hamilton-Jacobi theory is often used in celestial mechanics to transform the considered problem to variables in which the dynamics appear in a much simpler form compared with the initial one. It is often asked for a canonical transformation which makes the new momenta to be constant (maybe constantly equal to zero) and coordinates which are linear functions of time. The generating function (let us label it S), as it is the case in our article, has to be found accordingly in a perturbative manner. It is of the physical dimension of an action and is, therefore, simply called the “*action*” in the subsequent lines.

A. Action and angle variables: basics

In this section we will derive the action-angle variables for the equal-mass-two-spin system. Before that, let us state why those kinds of variables are so useful. Throughout this section, the Einstein summation convention is *not employed*. The starting point is the Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = -\frac{\partial S}{\partial t}. \quad (4.1)$$

Let us suppose that the energy, or the value of the Hamiltonian, is conserved. Then, taking this as input for the HJ equation, the action can be separated as follows:

$$S = -E t + W(q), \quad (4.2)$$

The function $W(q)$ of spatial coordinates is called the characteristic function. There exists a number of conserved quantities $\alpha_1, \alpha_2, \dots$ – we may, for example, define $\alpha_1 = E, \alpha_2 = L \dots$ for a system in which the magnitude of the orbital angular momentum and further momenta are also conserved. The following quantities, called *action variables*, turn out to be interesting when related to W ,

$$J_k := \frac{1}{2\pi} \oint p_k dq_k, \quad (4.3)$$

where the integral is meant for one complete orbit. Here, it holds $p_k = p_k(q_i, \alpha_i)$ for Staeckel systems. The p_k being q_k librational, these J_k do not depend on q_k any more and thus, one may express the J_k in terms of the α_i alone,

$$J_k = J_k(\alpha_1, \alpha_2, \dots). \quad (4.4)$$

If we turn these relations “inside-out”, giving $\alpha_i = \alpha_i(J_k)$, we obtain for the characteristic function

$$W = W(\mathbf{q}, \mathbf{J}). \quad (4.5)$$

Because the generating function is of that special type and $S : (p, q) \rightarrow (J, \omega)$, where the q are old coordinates and J are the new momenta, one computes the coordinate transformation according to

$$p_k = \frac{\partial W}{\partial q_k}, \quad (4.6)$$

$$w_k = \frac{\partial W}{\partial J_k}. \quad (4.7)$$

The Hamiltonian H , as it is conserved and identified with α_1 , is now a function of the J 's alone,

$$H = \alpha_1(\mathbf{J}). \quad (4.8)$$

The main frequencies can be obtained via

$$\dot{J}_k = -\frac{\partial H}{\partial w_k} = 0, \quad (4.9)$$

$$\dot{w}_k = \frac{\partial H}{\partial J_k} = \frac{\partial \alpha_1(\mathbf{J})}{\partial J_k} =: \nu_k. \quad (4.10)$$

In the subsequent lines we will present the calculation of essential action and angle variables and how to deal with degenerate systems. In the end, we will perform a transformation to variables related to the well-known Delaunay variables.

B. Separating the action: AA-Variables for the integrable system

Taking the Hamiltonian in the form of [19] and replacing all the momenta (especially: the spins) by derivatives of the action integral, we see that the spin parts are completely separable⁴.

$$\begin{aligned}
H_{\text{tot}} = & \frac{p_r^2}{2} - \frac{1}{r} + \frac{L^2}{2r^2} \\
& + c^{-2} \left\{ \frac{L^4(3\eta-1)}{8r^4} + \frac{L^2(3\eta-1)p_r^2 + 2}{4r^2} - \frac{L^2(\eta+3)}{2r^3} + \frac{1}{8}(3\eta-1)p_r^4 - \frac{(2\eta+3)p_r^2}{2r} \right\} \\
& + \frac{c^{-3}}{r^3} \left\{ \frac{(-J^2 + L^2 + S_G^2)(12\epsilon(S_2^2 - S_1^2) + S_G^2(4\epsilon^2 - 7))}{16S_G^2} - \frac{3\epsilon \sin(\phi_S) \sqrt{F_4(J, L, S_G)F_4(S_1, S_2, S_G)}}{4S_G^2} \right\} \\
& + c^{-4} \left\{ \frac{1}{16}(5(\eta-1)\eta+1)p_r^6 + \frac{L^6}{16r^6}(5(\eta-1)\eta+1) - \frac{L^4}{8r^5}(\eta(3\eta+20)-5) \right. \\
& + \frac{1}{r^4} \left(\frac{3}{16}L^4(5(\eta-1)\eta+1)p_r^2 + L^2 \left(4\eta + \frac{5}{2} \right) \right) + \frac{1}{4r^3} ((L^2(5-4\eta(\eta+5))p_r^2 - 3\eta-1)) \\
& \left. + \frac{1}{r^2} \left(\frac{3}{16}L^2(5(\eta-1)\eta+1)p_r^4 + \frac{1}{2}(11\eta+5)p_r^2 \right) + \frac{1}{r} \left(\frac{1}{8}(5-4\eta(2\eta+5))p_r^4 \right) \right\}, \tag{4.11}
\end{aligned}$$

Here, the functions F_4 are polynomials of the angular momentum magnitudes (also see Eqs. (5.13) and (5.14) of [19]),

$$F_4(J, L, S_G) := (J-L-S_G)(J+L-S_G)(J-L+S_G)(J+L+S_G), \tag{4.12}$$

$$F_4(S_1, S_2, S_G) := (S_1-S_2-S_G)(S_1+S_2-S_G)(S_1-S_2+S_G)(S_1+S_2+S_G). \tag{4.13}$$

The doubly underlined sin-term is an oscillatory term for the quasi-circular case only in the sense that, as one inserts the solution to the rest of the Hamiltonian, its average over one time period⁵ of ϕ_S is exactly zero. In the following, we will show how to include the sin-term (as a small deviation from the equal-mass limit) into the equal-mass solution (as the unperturbed problem). First, we have to find action and angle variables for the unperturbed problem. Secondly, with the help of these variables, we perform a canonical transformation that shifts the sin-term to the order $\mathcal{O}(\epsilon^2)$ of the mass difference parameter.

Structurally, this looks as follows:

- decomposition:
$$H = \underbrace{H^*}_{\text{integrable}} + \underbrace{H_{\text{SO}}^{\text{pert}}}_{\text{small}} \quad \text{with} \quad \mathcal{O}(H_{\text{SO}}^{\text{pert}}) = \epsilon^1.$$
- find AA variables (\mathbf{I}, \mathbf{w}) for H^* only:
$$H^* = H^*(\mathbf{I}) \quad \Rightarrow \quad H_{\text{SO}}^{\text{pert}} = H_{\text{SO}}^{\text{pert}}(\mathbf{I}, \mathbf{w})$$
- find generator:
$$H^*(\mathbf{I}) \rightarrow H^{*(1)}(\mathbf{I}'); \quad H_{\text{SO}}^{\text{pert}}(\mathbf{I}, \mathbf{w}) \rightarrow H_{\text{SO}}^{\text{pert}'}(\mathbf{I}', \mathbf{w}') \quad \text{with} \quad \mathcal{O}(H_{\text{SO}}^{\text{pert}'}) = \epsilon^2.$$

To provide more details for finding the AA variables first, we use the full-separation ansatz for the action S and the function W , namely Eq. (4.2), where E is the energy of the system which is negative in the bound-orbit case and $|E|$ the value of the binding energy that appears in the solution for the orbital elements a_r, e_r and so on; q are all the spatial coordinates, $q = \{r, \phi, \Upsilon, \phi_S, \dots\}$. We justify this separation ansatz below. The form of W reads

$$W(q) := W_r(r) + W_\phi(\phi) + W_\Upsilon(\Upsilon) + W_{\text{spin}}(q_{\text{spin}}), \tag{4.14}$$

where

$$W_{\text{spin}}(q_{\text{spin}}) := W_{\alpha_{S1}}(\alpha_{S1}) + W_{\alpha_{S2}}(\alpha_{S2}) + W_{\phi_S}(\phi_S). \tag{4.15}$$

⁴ Separable means that we can construct the action in terms of summands for the spin parts and other parts that are associated with the remainder with certain separate properties we do not specify here.

⁵ “Time period” is as valid as the term “period” alone because it holds $\phi_S(t - t_0) = \Omega_S t$ having $\Omega_S = \text{const.}$

Here, the α_{S_a} with $a \in \{1, 2\}$ are intrinsic rotation angles of the individual objects that do not appear explicitly in the Hamiltonian because of the absence of spin-spin and spin-squared interaction terms. The following discussion shows some details of the computation for the case $H_{\text{N+SO}}^{\text{no } \phi_S} := H_{\text{PM}}^{\text{N}} + H_{\text{SO}}^{\text{LO, no } \phi_S}$ without the ϕ_S -dependent part (the integrable part is what then remains), as we move to coordinates in which the 3-component of the orbital angular momentum \mathbf{L} is eliminated and only the scalar contribution L appears⁶. The extension to the 2PN Hamiltonian without the ϕ_S part is done in the same way and gives the same structure of terms. One also observes that the Hamiltonian does not depend on orientations such as Υ , ϕ_S , and as mentioned α_{S1} and α_{S2} , which means that the “old momenta” L, S_G, S_1 and S_2 are conserved and transformed into themselves (this part of the generating function being the identity transformation). One can therefore still write S_G and J instead of $W_{\phi_S}(\phi_S)$ and $W_{\Upsilon}(\Upsilon)$, respectively:

$$H_{\text{N+SO}}^{\text{no } \phi_S} = \frac{1}{2} \left(p_r^2 + \frac{L^2}{r^2} \right) - \frac{1}{r} + \frac{1}{c^3 r^3} (J^2 - L^2 - S_G^2) (2\eta + 3). \quad (4.16)$$

The Hamiltonian does not depend on the variable ϕ either and thus one can write down $W_{\phi}(\phi) = L\phi$. We may write down the above integrable part with the input of Eq. (4.2) and obtain

$$W_r'^2 + \frac{1}{r^2} L^2 - \frac{2}{r} + \frac{2}{c^3 r^3} (J^2 - L^2 - S_G^2) (2\eta + 3) = 2E, \quad (4.17)$$

(a prime in W_r' means partial derivative with respect to r) from where one (formally) easily extracts the W_r part as an integral over a square-root. The explicit computation of the W_r part is discussed in Appendix A.

1. Results

The spin-orbit Hamiltonian yields the following action variables,

$$\begin{aligned} J_r = & -L - \frac{1}{\sqrt{2}\sqrt{|E|}} + c^{-2} \left(\frac{3}{L} - \frac{(\eta - 15)\sqrt{|E|}}{4\sqrt{2}} \right) \\ & - \frac{c^{-3}(J^2 - L^2 - S_G^2)(3\sqrt{1-4\eta}(S_1 - S_2)(S_1 + S_2) + (2\eta + 3)S_G^2)}{8L^3 S_G^2} \\ & - \frac{c^{-4}(\sqrt{2}L^3(3\eta(\eta + 10) + 35)|E|^{3/2} + 96L^2(5 - 2\eta)|E| + 80(2\eta - 7))}{64L^3}, \end{aligned} \quad (4.18a)$$

$$J_{\phi} = L, \quad (4.18b)$$

$$J_{\Upsilon} = J, \quad (4.18c)$$

$$J_S = S_G, \quad (4.18d)$$

where the subscripts on the left hand sides denote the coordinate over which has been integrated, with the exception of the subscript S for the spin part for reasons of beauty⁷. We observe

$$(J_r + J_{\phi})^2 - a_r = \mathcal{O}(c^{-2}). \quad (4.19)$$

Within perturbation theory, the Hamiltonian (energy) can be expressed as

$$\begin{aligned} -H^*(\mathbf{J}) = & \frac{1}{2(J_r + J_{\phi})^2} \left[1 + \frac{c^{-2}}{(J_r + J_{\phi})^2} \left(\frac{\eta + 9}{4} + \frac{6J_r}{J_{\phi}} \right) \right. \\ & - \frac{c^{-3}}{(J_r + J_{\phi})} \left(\frac{(J_{\Upsilon}^2 - J_{\phi}^2 - J_S^2)(3\sqrt{1-4\eta}(S_1 - S_2)(S_1 + S_2) + (2\eta + 3)J_S^2)}{4J_{\phi}^3 J_S^2} \right) \\ & \left. + \frac{c^{-4}}{(J_r + J_{\phi})^4} \left(\frac{5(7 - 2\eta)J_r^3}{2J_{\phi}^3} + \frac{3(53 - 10\eta)J_r^2}{2J_{\phi}^2} - \frac{9(\eta - 6)J_r}{J_{\phi}} + \frac{1}{8}((\eta - 7)\eta + 81) \right) \right]. \end{aligned} \quad (4.20)$$

⁶ This procedure can also be performed in general spherical coordinates where the elimination has not been done so far. Such a discussion for the Newtonian case alone can be found in the books [45] and [37].

⁷ We assume that the orbital angular momentum \mathbf{L} fulfils $L_z = \mathbf{e}_z \cdot \mathbf{L} > 0$, see [32] for details.

We see that J_Υ does not appear in the point-mass parts, and through Newtonian order only, J_r and J_ϕ are degenerate. We next see what happens when we examine a removal of possible degeneracies, i.e. a transformation to variables that absorb conditions of degeneracy.

C. Degenerated systems: Delaunay variables for the spin-orbit Hamiltonian

If a system of n degrees of freedom has an m -fold degeneracy, meaning that the first m frequencies are not linearly independent in the sense

$$\sum_{i=1}^m n_{\alpha i} \omega_i = 0, \quad \alpha: \text{labeling the } \alpha^{\text{th}} \text{ condition}, \quad (4.21)$$

one can construct a generator of type 2 – in the sense of common literature on theoretical mechanics – F_2 : $(\omega \rightarrow \bar{\omega}, J \rightarrow I)$ of the form

$$F_2(\omega_i, I_i) = \sum_{k=1}^m \sum_{i=1}^n n_{ki} \omega_i I_k + \sum_{k=m+1}^n \omega_k I_k \quad (4.22)$$

where n_{ki} is a coefficient of the k^{th} degeneracy condition to connect the angle variables with index i , such as for a fictitious set of variables ω^*

$$k = 1: \quad n_{11}\omega_1^* + n_{13}\omega_3^* + \dots = 0, \quad (4.23a)$$

$$k = 2: \quad n_{21}\omega_1^* + n_{22}\omega_2^* + \dots = 0, \quad (4.23b)$$

$$\dots \text{and so on}, \quad (3.23n)$$

resulting in

$$\left. \begin{aligned} \bar{\omega}_k &= \frac{\partial F_2}{\partial I_k} = \begin{cases} \sum_{i=1}^n n_{ki} \omega_i & \text{for } k = 1, \dots, m \\ \omega_k & \text{for } k = m+1, \dots, n \end{cases} \\ J_i &= \frac{\partial F_2}{\partial \omega_i} = \sum_{k=1}^m n_{ki} I_k + \sum_{k=m+1}^n \delta_{ki} I_k \end{aligned} \right| \quad \bar{\omega}_k = \frac{\partial H}{\partial I_k} = 0 \text{ for } k=1\dots m \quad (4.24)$$

$$\Rightarrow H = H(\mathbf{I})$$

In the case of a Newtonian binary compact object we observe⁸

$$\omega_r - \omega_\phi = 0 \quad \Rightarrow \quad n_{1\phi} = -1, \quad n_{1r} = 1, \quad n_{1\Upsilon} = 0; \quad (4.25)$$

so our generating function will look as follows,

$$F_2(\omega, \mathbf{I}) = (\omega_\phi - \omega_r) I_1 + \omega_r I_2 + \omega_\Upsilon I_3 + \omega_S I_4. \quad (4.26)$$

From Eq. (4.24) the transformation of the momenta and coordinates yields

$$\begin{aligned} J_r &= I_2 - I_1, & w_1 &= w_\phi - w_r, \\ J_\phi &= I_1, & w_2 &= w_r, \\ J_\Upsilon &= I_3, & w_3 &= w_\Upsilon, \\ J_S &= I_4, & w_4 &= w_S, \end{aligned} \quad (4.27)$$

The transformation from old to new momenta is simply obtained by inversion of the above system. The total integrable Hamiltonian H^* , written in terms of the new \mathbf{I} , then reads

$$-H^*(\mathbf{I}) = \frac{1}{2I_2^2} + \frac{c^{-2}}{2I_2^2} \left(\frac{\eta - 15}{4I_2^2} + \frac{6}{I_1 I_2} \right) + c^{-3} \frac{(I_1^2 - I_3^2 + I_4^2) ((2\eta + 3)I_4^2 + 3\sqrt{1 - 4\eta}(S_1 - S_2)(S_1 + S_2))}{8I_1^3 I_2^3 I_4^2}$$

⁸ Newtonian binaries do not suffer periastron shift, therefore the radial period is the same as the angular, see Eq. (4.20).

$$+ \frac{c^{-4}}{2I_2^2} \left(\frac{5(7-2\eta)}{2I_1^3 I_2} + \frac{3(4\eta-35)}{2I_1 I_2^3} + \frac{(\eta-15)\eta+145}{8I_2^4} + \frac{27}{I_1^2 I_2^2} \right). \quad (4.28)$$

This is the integrable part as a function of what is known as Delaunay variables $(I_1, I_2, I_3; \omega, \ell, \Upsilon)$ and their extension of the spin magnitudes and total angular momentum, see below for explanation. Taking the action variables in Ref. [46] which differ by the re-definition $I_2 \rightarrow i_3$, $I_1 \rightarrow i_2$, this exactly reproduces those authors' result through 1PN, see their Eq. (3.13). Again, this labelling discrepancy results from the missing degeneracy in our Hamiltonian that would be present if we used an unspecified⁹ frame for a derivation instead. To make contact with Vinti's notation [37] of Delaunay's variables, marked with subscript “ D ”, let us give the following (Newtonian) relations, which will be needed for the first-order perturbation generator:

$$L_D = \sqrt{a} = I_2, \quad \ell_D = \mathcal{N}(t + \beta_1), \quad (4.29)$$

$$G_D = |L| = I_1, \quad g_D = \omega, \quad (4.30)$$

$$H_D = L_Z + S_Z = J_\Upsilon = I_3, \quad h_D = \Upsilon, \quad (4.31)$$

$$S_D = S_G, \quad s_D = \phi_S, \quad (4.32)$$

$$\Sigma_{1D} = S_1, \quad \sigma_{1D} = \alpha_{s1}, \quad (4.33)$$

$$\Sigma_{2D} = S_2, \quad \sigma_{2D} = \alpha_{s2}, \quad (4.34)$$

β_1 being the linear-in-time coordinate function that is associated with the constant “momentum” α_1 , and ω as the argument of the pericenter. The $(\sigma_{aD}, \Sigma_{aD})$ section of the above block of variables is not present in the Newtonian case and has been added to complete the phase space. That means that the variable ϕ has been removed by means of the Newtonian degeneracy condition. Taking Eqs. (3.14) and (3.15) of [46],

$$\mathcal{N} := \frac{\partial H^*}{\partial I_2} = \frac{\partial H^*}{\partial L_D}, \quad (4.35)$$

$$k \mathcal{N} = \frac{\partial H^*}{\partial I_1} = \frac{\partial H^*}{\partial G_D}, \quad (4.36)$$

we obtain for the periastron advance parameter k

$$k := \frac{\Phi - 2\pi}{2\pi} = \frac{3}{c^2 I_1^2} = \frac{3}{c^2 L^2} + \mathcal{O}(c^{-3}), \quad (4.37)$$

which is a well-known result. Further, the relations $e = \sqrt{1 - 2EL^2}$ and $E = \frac{-1}{2a}$ hold – again only in the Newtonian case – such that

$$e^2 = 1 - \left(\frac{G_D}{L_D} \right)^2 + \mathcal{O}(c^{-2}). \quad (4.38)$$

We are aware that in the quasi-circular limit, G_D and L_D are degenerate. In that case, one is forced to transform to another set of variables that incorporates this degeneracy, for example the Poincaré elements as pointed out in the notes of Howison and Meyer [47] or to the approach in [19]. However, our calculation starts from the eccentric case, meaning that the startup to the solution is not evaluated on the circular orbit. One can deal generally with an eccentric system and let, finally on the solution level, let e tend to zero.

Subsequently, we will present basics of canonical perturbation theory for the action and angle variables and the application to the eccentric spin-orbit problem. This perturbation theory aims to find a generator for a canonical transformation that shifts contributions of the total interaction Hamiltonian which have oscillatory dependencies on phase space coordinates (called ϕ here) to a higher order of the small expansion parameter ϵ , resulting in a new Hamiltonian that only depends on the transformed momenta. As for the circular-orbit case in Ref. [19], the mass difference function will be chosen to be the mentioned smallness parameter.

⁹ “Unspecified” means that we would take a general direction of \mathbf{L} and look for the spherical coordinate contributions, not only the planar problem in the unperturbed Newtonian case.

V. HAMILTON-JACOBI PERTURBATION THEORY WITH ACTION-ANGLE VARIABLES

As it could be seen in Eq. (4.11), the Hamiltonian contains a term that depends on the spin orientation phase ϕ_S which is of the order $\mathcal{O}(\epsilon^1)$ and not included in the known solutions. Writing the total Hamiltonian in terms of the Delaunay elements in which the unperturbed Hamiltonian only depends on the momenta and with the help of a further canonical transformation, we like to shift that expression to order $\mathcal{O}(\epsilon^2)$. Below, we list the basic properties of such a general canonical transformation.

The task is to solve the Hamilton-Jacobi equation

$$H(\phi, \partial_\phi S(\phi, \mathbf{I}')) = H'(\mathbf{I}'), \quad (5.1)$$

where the right hand side only depends on the new momenta \mathbf{I}' , not on the angles ϕ , perturbatively, although the existence of a solution S may not be guaranteed. We expand the generator S around the identity transformation ($\phi \rightarrow \phi, \mathbf{I} \rightarrow \mathbf{I}' = \mathbf{I}$) in powers of the perturbation parameter ϵ and set

$$S(\phi, \mathbf{I}') = S_0 + \epsilon S_1(\phi, \mathbf{I}') + \epsilon^2 S_2(\phi, \mathbf{I}') + \mathcal{O}(\epsilon^3), \quad (5.2)$$

$$S_0(\phi, \mathbf{I}') = \phi \cdot \mathbf{I}' \quad (= \text{identity transformation}), \quad (5.3)$$

$$S_1(\phi, \mathbf{I}') \dots \text{ to be found,} \quad (5.4)$$

which gives, up to first order in ϵ ,

$$\begin{aligned} H_0(\partial_\phi(S_0 + \epsilon S_1)) + \epsilon H_1(\phi, \partial_\phi(S_0 + \epsilon S_1)) &= H'(\mathbf{I}'), \\ H_0(\mathbf{I}') + \epsilon \partial_{\mathbf{I}} H_0(\mathbf{I})|_{\mathbf{I}=\mathbf{I}'} \partial_\phi(S_1) + \epsilon H_1(\phi, \mathbf{I}') &= H'(\mathbf{I}'), \end{aligned} \quad (5.5)$$

where we have used $H_0 = H_0(\mathbf{I})$. Subtracting $H'(\mathbf{I}')$ gives zero on the right hand side. The resulting relation can be fulfilled only if each coefficient of powers of ϵ is equivalent to 0, i.e.

$$0: \quad H_0(\mathbf{I}') \stackrel{!}{=} H'(\mathbf{I}'), \quad (5.6)$$

$$\begin{aligned} 1: \quad \partial_{\mathbf{I}} H_0(\mathbf{I})|_{\mathbf{I}=\mathbf{I}'} \partial_\phi(S_1) &\stackrel{!}{=} -H_1(\phi, \mathbf{I}') \\ \Rightarrow \quad \omega \cdot \partial_\phi S_1(\phi, \mathbf{I}') &= -H_1(\phi, \mathbf{I}') \end{aligned} \quad (5.7)$$

Here, $\omega = \omega(\mathbf{I}')$ holds.¹⁰ To solve this, we make the Fourier ansatz

$$S_1(\phi) = \sum_{k_1, \dots, k_j = -\infty}^{\infty} S_{1\mathbf{k}} e^{i\mathbf{k} \cdot \phi} \quad (5.8)$$

and accordingly for H_1 . Here, \mathbf{k} is a multi-index, and $\mathbf{k} \cdot \phi = \sum_i k_i \phi_i$. With this input and Eq. (5.7) we obtain

$$\partial_{\phi_j} S_1(\phi) = i \sum_{\mathbf{k}} S_{1\mathbf{k}} k_j e^{i\mathbf{k} \cdot \phi}, \quad (5.9)$$

$$\sum_{\mathbf{k}} \left\{ i \omega \cdot \mathbf{k} S_{1\mathbf{k}}(\mathbf{I}') + H_{1\mathbf{k}}(\mathbf{I}') \right\} e^{i\mathbf{k} \cdot \phi} = 0. \quad (5.10)$$

This has to hold for arbitrary ϕ , meaning that all the \mathbf{k} -coefficients vanish:

$$S_{1\mathbf{k}}(\mathbf{I}') = i \frac{H_{1\mathbf{k}}(\mathbf{I}')}{\omega \cdot \mathbf{k}}. \quad (5.11)$$

The reader should be warned that the inner product in Eq. (5.11) may vanish for special systems. We have to show that, for our problem, the system does not fulfil any exact degeneracy condition. The Hamiltonian H_0 will be identified with H^* and H_1 with H^{pert} in the subsequent sections.

¹⁰ The ω in above equation is computed using the derivative with respect to the unprimed variables, thereafter replacing all variables \mathbf{I} by the primed ones \mathbf{I}' without using the variable transformation which, anyway, still has to be obtained.

A. The perturbing Hamiltonian: Series expansion around the circular and equal-mass case

In this section we extract the oscillatory parts of the full spin-orbit problem. In [19] we saw that there exist oscillatory terms for the circular orbit case. In addition, eccentricity will also create oscillations. Therefore, we expand the full Hamiltonian around the equal-mass case (here: to the first power of ϵ) and, as well, present it in powers of eccentricity e through fourth order.

The sin-part of the Hamiltonian symbolically reads

$$H_{\text{SO}}^{\text{pert}} = \epsilon \frac{1}{r^3} \mathcal{G}(\mathbf{X}_{\text{ang}}) \sin \phi_S, \quad (5.12)$$

where \mathcal{G} is a function of the angular momenta amplitudes \mathbf{X}_{ang} solely, see Eq. (4.11). Our task is now to express this Hamiltonian in terms of the Delaunay variables from the previous section with the help of an eccentricity expansion around the initial solution. The “solution” to the unperturbed problem will be that for the case $m_1 = m_2$ and $e \neq 0$, see Eqs. (3.1) [29].

We may lend help from Ref. [48], where inverse powers (n) of r are expanded first in harmonics of \mathcal{E} and afterwards in ℓ_D which is the desired result. We already know that it holds for $A := (1 - e \cos \mathcal{E})$

$$A^{-n} = \sum_{j \geq 0} \mathcal{A}_j^{(n)} \cos(j \ell_D) \quad (5.13)$$

where $\mathcal{A}_j^{(n)}$ is a relatively complicated function of the eccentricity, factorials and Bessel functions of the summation index j (also see Eq. (37) of [36], also the standard material in [49] and, for further investigations on a post-circular expansion for gravitational wave generation in the Newtonian case, Ref. [9]). We may expand \mathcal{A}^{-n} to, say, fourth order¹¹ in e :

$$\begin{aligned} A^{-n} = & 1 + e^2 \left(\frac{n^2}{4} - \frac{n}{4} \right) + e^4 \left(\frac{n^4}{64} + \frac{n^3}{32} - \frac{n^2}{64} - \frac{n}{32} \right) + \cos(\ell_D) \left(e^3 \left(\frac{n^3}{8} + \frac{n^2}{8} - \frac{3n}{8} \right) + en \right) \\ & + \cos(2\ell_D) \left(e^4 \left(\frac{n^4}{48} + \frac{n^3}{8} - \frac{n^2}{48} - \frac{11n}{24} \right) + e^2 \left(\frac{n^2}{4} + \frac{3n}{4} \right) \right) + e^3 \cos(3\ell_D) \left(\frac{n^3}{24} + \frac{3n^2}{8} + \frac{17n}{24} \right) \\ & + e^4 \cos(4\ell_D) \left(\frac{n^4}{192} + \frac{3n^3}{32} + \frac{95n^2}{192} + \frac{71n}{96} \right) \end{aligned} \quad (5.14)$$

to read-off the coefficients $\mathcal{A}_j^{(n)}$. **An important remark:** This has been done to Newtonian order only. A generalisation including 1PN terms in the perturbing function would let us distinguish the “radial” and “time” eccentricities e_r and e_t appearing in the solution $\ell_D(\mathcal{E})$ and the expression $A(r)$ to be combined in an extension of our Eq. (5.14). Going further to 2PN order would mean to include Eqs. (102)–(110) of Ref. [50] and an expansion of regularised hyper-geometric functions to some order of e_t . Our aim is to deliver the knowledge for the leading order, so we sketch the way for the Newtonian Kepler equation only.

B. Examining the perturbation and the generator in the Fourier domain

With these inputs, we can easily express the perturbing Hamiltonian $H_{\text{SO}}^{\text{pert}}$ as

$$H_{\text{SO}}^{\text{pert}} = \frac{1}{L_D^6} \epsilon \mathcal{G}(\mathbf{X}_{\text{ang}}) \sin(\phi_S) \sum_{j=0}^{\infty} \mathcal{A}_j^{(3)}(G_D, L_D) \cos(j \ell_D). \quad (5.15)$$

In expanded and full-canonical form (except of the Newtonian e appearance which can be avoided by using Eq. (4.38)), it reads

$$H_{\text{SO}}^{\text{pert}} = -\frac{3c^{-3}\epsilon \sin(\phi_S)}{4L_D^6 S_G^2} \left\{ 1 + \frac{3e^2}{2} + \frac{15e^4}{8} + \left(\frac{27e^3}{8} + 3e \right) \cos(\ell_D) + \left(\frac{7e^4}{2} + \frac{9e^2}{2} \right) \cos(2\ell_D) \right.$$

¹¹ A general expression for arbitrary n seems to be obtainable, but has not been found yet. Its coefficients for finite n are easy to be calculated manually.

$$\left. + \frac{53}{8} e^3 \cos(3\ell_D) + \frac{77}{8} e^4 \cos(4\ell_D) \right\} \sqrt{F_4(J, L, S_G) F_4(S_1, S_2, S_G)}. \quad (5.16)$$

Therefore, we need Fourier transformations of

$$\mathcal{F}_k(\sin(a\phi_S), \phi_S) = \frac{1}{2i} (\delta_{a,k} - \delta_{a,-k}), \quad (5.17a)$$

$$\mathcal{F}_k(\cos(a\ell_D), \ell_D) = \frac{1}{2} (\delta_{a,k} + \delta_{a,-k}), \quad (5.17b)$$

with integer k , where $\mathcal{F}(f(q), q, k)$ is the k^{th} Fourier coefficient of the function f ,

$$f(q) = \sum_{j=-\infty}^{\infty} \mathcal{F}_j(f(q), q) e^{ijq}, \quad (5.18)$$

$$\mathcal{F}_k(f(q), q) := \frac{1}{2\pi} \int_0^{2\pi} f(q) \exp(-ikq) dq. \quad (5.19)$$

As the zeroth-order Hamiltonian is independent of ω_j , defining the new main angular velocities ω_i with respect to the Delaunay variables according to

$$\omega_j = \left. \frac{\partial H_0(\mathbf{I})}{\partial I_j} \right|_{\mathbf{I}=\mathbf{I}'}, \quad j = 1, \dots, 4, \quad (5.20)$$

yields

$$\begin{aligned} S_{1k}(\mathbf{I}') &= \frac{1}{(\mathbf{k} \cdot \boldsymbol{\omega})} \frac{1}{2} \frac{c^{-3}\epsilon}{L_D^6 S_D^2} \sqrt{F_4(J, L, S_G) F_4(S_1, S_2, S_G)} (\delta_{1,-k_4} - \delta_{1,k_4}) \times \\ &\quad \left(\frac{3}{4} + \frac{9}{8} e (\delta_{1,-k_1} + \delta_{1,k_1}) \right. \\ &\quad + \frac{9}{16} e^2 (3\delta_{2,-k_1} + 3\delta_{2,k_1} + 2) \\ &\quad + \frac{3}{64} e^3 (27\delta_{1,-k_1} + 27\delta_{1,k_1} + 53(\delta_{3,-k_1} + \delta_{3,k_1})) \\ &\quad \left. + \frac{3}{64} e^4 (28\delta_{2,-k_1} + 28\delta_{2,k_1} + 77\delta_{4,-k_1} + 77\delta_{4,k_1} + 30) \right) \Big|_{\mathbf{I} \rightarrow \mathbf{I}'} + \mathcal{O}(e^5) \end{aligned} \quad (5.21)$$

in the Fourier representation analogous to Eq. (5.8).¹²

The solution to the perturbed problem now consists of performing the coordinate transformation explicitly; that means a transformation to the new momenta and new phase coordinates. Since the generator depends on the old $\boldsymbol{\phi}$ and new \mathbf{I}' , use has to be made of the relations

$$\mathbf{I} = \frac{\partial S}{\partial \boldsymbol{\phi}}, \quad (5.22)$$

$$\boldsymbol{\phi}' = \frac{\partial S}{\partial \mathbf{I}'}. \quad (5.23)$$

The first set of equations is to be inverted for \mathbf{I}' , then the resulting relations have to be inserted in the second set to eliminate \mathbf{I}' in favour of the old \mathbf{I} to finally obtain

$$\boldsymbol{\phi}' = \boldsymbol{\phi}'(\boldsymbol{\phi}, \mathbf{I}), \quad (5.24)$$

$$\mathbf{I}' = \mathbf{I}'(\boldsymbol{\phi}, \mathbf{I}). \quad (5.25)$$

For a second transformation, the full information to the solution (for the new Hamiltonian in the new coordinates) has to be found. The reader should be aware that, going to the n^{th} transformation, all terms to order n have to be kept in the generator approximation process until the end. That means that also the generating function S itself has to be Taylor expanded to order $\mathcal{O}(\epsilon^n)$.

The convergence of the Fourier series Eq. (5.8), also having a hidden dependence on the higher-order-in- e contribution that provides the higher \mathbf{k} terms has to be discussed. This can be done with the help of the *Kolmogoroff, Arnold and Moser* (KAM) theory.

¹² The reader is reminded of the fact that this relation is not a time-Fourier representation.

C. Some remarks about the non-degeneracy of the Delaunay frequencies ω_j

The KAM theory [51] states that for sufficiently non-degenerate systems (in classical lectures, other conditions than $\sum n_{\alpha i} \omega_i \neq 0$ are given; the strong nonresonance: the existence of constants $\alpha > 0$ and $\tau > 0$ such that $|\langle k\omega \rangle| \geq \frac{\alpha}{|k|^\tau}$ for all $0 \neq k \in \mathbb{Z}^n$ with $|k| = \sum_i |k_i|$), the series expansion (5.11) converges. In Arnold's book [52], p. 408, the following condition

$$\det \left| \frac{\partial^2 H_0}{\partial \mathbf{I}^2} \right| \neq 0, \quad (5.26)$$

was provided which guarantees conservation of most invariant tori under small perturbations. We like to state that, although we insert the Newtonian-order solution into the perturbation generator because of our PN truncation, what we like to perturb is not the Newtonian solution but the equal-mass 1PN SO + 2PN PM solution. In that context, our approximation is too crude to see the periastron advance and spin precession effects in the generator itself, so what will be required in an extension to higher PN orders of this generator in order to include the “missing” dynamics. Although the non-degeneracy condition (5.26) is not fulfilled in the Newtonian case (the denominator in (5.21) then anyway would only contain one single frequency rather than a summation), it definitely is so in the PN case. Therefore, the general relativistic solutions are “much more stable” with respect to perturbations.

It is, for the time being, unclear (i), how large the mass difference and (ii), how large the binding energy of the system is allowed to be (possibly generating degenerate frequencies at some point of the evolution downwards inspiral) before the deformed tori are finally destroyed.

VI. CONCLUSIONS

In this article, we presented a first-order solution to the eccentric two-body problem with spin-orbit coupling having slightly different masses. We expressed the solution to the well-known equal-mass solution in terms of Delaunay-type variables. With the help of these variables, we constructed a canonical transformation which shifts the perturbing Hamiltonian part, characterised by the sin-function of the spin orientation phase ϕ_S and being of first order of the mass-difference function ϵ , to second order where it may also contain $\cos \phi_S$ terms.

As a task to remain for a future publication it has to be found out how large the mass difference is allowed to be before the deformed KAM tori are destroyed. Further, one has to take into account the next-to leading order of the spin-orbit interaction, which means that in the Fourier expansion of the inverse distance it has to be distinguished between e_t and e_r , which modifies the solution at higher orders of inverse c .

A remark on Delaunay elements in higher orders of c : We computed the quantity J_r as a definite integral over the radial variable. In order to express the time t as a function of the variable ℓ_D in higher orders of c , we may use a generating function of the form

$$W = L\phi + J\Upsilon + \int_{r_+}^r f_r(r') dr' + W_{\text{spin}}, \quad (6.1)$$

where r_+ denotes the radial distance at the periastron and $f(r)$ is constructed in such a way that the new variable ℓ_D is directly related to the time t as a derivative of W and closely related to the Kepler equation (see standard texts on Delaunay elements, e.g. [37], and also the quasi-Keplerian parameterisation for higher PN orders, for example [30, 44, 53]). Note that L is the orbital angular momentum and to be distinguished from the energy-related Delaunay element L_D . We could use the Newtonian relations from common literature (which did not have to be re-calculated) in the current article, but the above relation has to be taken care of in a further development. It may turn out that, therefore, not much effort or new quality of calculation has to be considered to obtain the higher PN-order result.

Finally, two more problems are remaining in this arena. The first one is to tackle spin-squared and spin-spin interactions. Those Hamiltonians have a simple appearance in the coordinate-independent form, but being expressed in terms of the Delaunay-type (or Hill-type) variables or those in [19], they get complicated in comparison to spin-orbit interactions. This circumstance deserves a careful consideration. The second one is the treatment of radiation reaction, where it is currently unclear how to combine the radiation interaction terms and the eccentricity vs. unequal-mass precession in reasonable order for an analytic consideration.

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Appendix A: A short excursion to the contour integration for p_r

The integral for J_r can be computed by applying the method of residues. The integration is running from r_1 to r_2 and back, defining r_1 to be the inner and r_2 to be the outer boundary, $r_1 < r_2$, see Fig 1. These two points represent the boundary of a branch cut in the complex r -plane. On the journey from r_1 to r_2 , p_r is positive, and backwards negative. Thus, a single integration from r_1 to r_2 can be split into $\frac{1}{2}$ times an integration above plus one below the real axis, taking into account the change of the signs when changing the direction of the path. What follows is an expansion of the integration to the whole real axis. There are only 2 singular points, namely 0 and ∞ . The sign of the square-root is “-” for $r < r_1$ and is “+” for $r > r_2$. Let $f(r)$ denote the radicand in p_r . Then the final result for J_r is (see the rotation directions and the signs of the radicand to be taken!)

$$J_r = \frac{1}{2\pi} \times 2\pi i \left(\text{Res} \left(\sqrt{f}, r = 0 \right) - \text{Res} \left(\sqrt{f}, r \rightarrow \infty \right) \right). \quad (\text{A1})$$

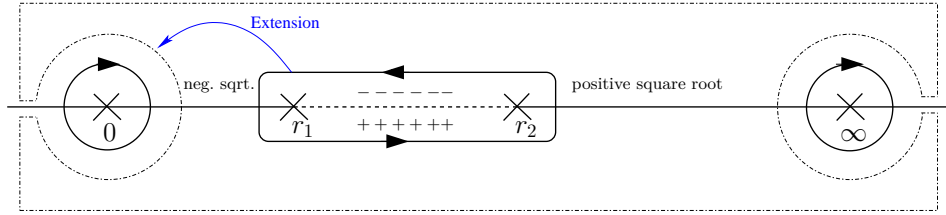


FIG. 1: Contour integral (also see Refs. [45] and [35]) for the application of the method of residues. The values $r = 0$ and $r = \infty$ are the only singular points. Below the real axis, the path is towards the apastron r_2 and thus the square-root has positive sign – a closed path computation is then possible. The dashed line is an intermediate step in deforming the contour in such a way that $r = 0$ and $r = \infty$ are the only excluded points.

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